CONSTANT COEF'S HOMOGENEOUS

4.1 + 4.3

CONSIDER $y''-3y'+2y=0$ ← y, y', y'' LOOK LIKE EACH OTHER EXCEPT POSSIBLY FOR CONSTANT MULTIPLES

GUESS $y=e^{rx}$ $r\in\mathbb{R}$

$y'=re^{rx}$

$y''=r^2e^{rx}$

$$y''-3y'+2y=r^2e^{rx}-3re^{rx}+2e^{rx}=0$$

$$(r^2-3r+2)e^{rx}=0$$

$$\underbrace{(r-1)(r-2)}_{=0}\underbrace{e^{rx}}_{\neq 0 \text{ FUNCTION}}=0$$

$r=1$ OR 2

$y=e^x$ OR e^{2x}

CHECK: $y=e^x$

$y'=e^x$ $\quad y''-3y'+2y=e^x-3e^x+2e^x=0$ ✓

$y''=e^x$

$y=e^{2x}$

$y'=2e^{2x}$ $\quad y''-3y'+2y=4e^{2x}-6e^{2x}+2e^{2x}=0$ ✓

$y''=4e^{2x}$

$y=e^x$ AND $y=e^{2x}$ ARE BOTH SOL'NS OF

$y''-3y'+2y=0$

IE. IF $L[y]=y''-3y'+2y$

THEN $L[e^x]=0$ AND $L[e^{2x}]=0$

AND SINCE L IS A L.D.O.

$y''+p(x)y'+q(x)y$
ARE LDO
FOR ALL p, q

$L[c_1e^x+c_2e^{2x}]$

$=c_1L[e^x]+c_2L[e^{2x}]$

$=0$

IE. $c_1e^x+c_2e^{2x}$ IS A SOL'N OF $y''-3y'+2y=0$

FOR ALL $c_1, c_2 \in \mathbb{R}$

DOES THIS COVER ALL SOL'NS OF $y''-3y'+2y=0$

4.1 EXISTENCE + UNIQUENESS FOR LINEAR ODE (E+U)

IF $p(x), q(x)$ AND $g(x)$ ARE CONTINUOUS ON (a,b)

AND $x_0 \in (a,b)$

THEN THE IVP $y'' + p(x)y' + q(x)y = g(x)$,

$y(x_0) = y_0,\ y'(x_0) = y_1$

HAS A UNIQUE SOL'N WHICH IS VALID ON (a,b)

$y'' - 3y' + 2y = 0$

$p(x) = -3,\ q(x) = 2,\ g(x) = 0$ ARE CONT. ON $(-\infty, \infty)$

BY E+U, IVP $y'' - 3y' + 2y = 0,\ y(x_0) = y_0,\ y'(x_0) = y_1$

IS GUARANTEED TO HAVE A UNIQUE SOL'N

IF $c_1 e^{x} + c_2 e^{2x}$ COVERS ALL SOL'NS OF $y'' - 3y' + 2y = 0$

THEN FOR ANY CHOICES OF x_0, y_0, y_1, THERE MUST BE CORRESPONDING VALUES OF c_1, c_2

$y = c_1 e^{x} + c_2 e^{2x}$ $\quad y' = c_1 e^{x} + 2c_2 e^{2x}$

$y(x_0) = c_1 e^{x_0} + c_2 e^{2x_0} = y_0$ ①

$y'(x_0) = c_1 e^{x_0} + 2c_2 e^{2x_0} = y_1$ ②

CAN WE FIND c_1, c_2 REGARDLESS OF WHICH VALUES OF x_0, y_0, y_1 WE START WITH?

$c_2 e^{2x_0} = y_1 - y_0$ ② − ①

$c_2 = \frac{y_1 - y_0}{e^{2x_0}} \in \mathbb{R}$ FOR ALL x_0, y_0, y_1

$c_1 e^{x_0} = 2y_0 - y_1$ $2 *$ ① − ②

$c_1 = \frac{2y_0 - y_1}{e^{x_0}} \in \mathbb{R}$ FOR ALL x_0, y_0, y_1

SO $y'' - 3y' + 2y = 0$, $y(x_0) = y_0$, $y'(x_0) = y_1$ HAS SOL'N

$$y = \frac{y_1 - y_0}{e^{2x_0}} e^{2x} + \frac{2y_0 - y_1}{e^{x_0}} e^{x} = c_2 e^{2x} + c_1 e^{x}$$

FOR ALL $x_0, y_0, y_1 \in \mathbb{R}$

IE. ALL SOL'NS OF $y'' - 3y' + 2y = 0$ HAVE FORM $\underbrace{y = c_1 e^{x} + c_2 e^{2x}}_{\text{GENERAL SOL'N OF } y'' - 3y' + 2y = 0}$

TH'M:

IF y_1, y_2 ARE SOL'NS OF $y'' + p(x)y' + q(x)y = 0$

THEN $y = C_1 y_1 + C_2 y_2$ IS THE GENERAL SOL'N

IFF (IF AND ONLY IF) $\underset{\text{WRONSKIAN}}{W[y_1, y_2]} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \neq \underset{\text{0 FUNCTION}}{0}$

$y_1 = e^x$ AND $y_2 = e^{2x}$ WERE SOL'NS OF $y'' - 3y' + 2y = 0$

$$W[e^x, e^{2x}] = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x \cdot 2e^{2x} - e^x \cdot e^{2x} = 2e^{3x} - e^{3x}$$
$$= e^{3x} \neq 0$$

SO, $y = C_1 e^x + C_2 e^{2x}$ IS THE GENERAL SOL'N OF $y'' - 3y' + 2y = 0$

$C_1 y_1 + C_2 y_2$ IS CALLED A LINEAR COMBINATION OF y_1, y_2

4.3

CONSTANT COEF HOMOGENEOUS

SOLVE $3y'' - 4y' + y = 0$ $\quad y, y', y''$ ARE LIKE EACH OTHER

GUESS $y = e^{rx}$

$y' = re^{rx}$

$y'' = r^2 e^{rx}$

$$3y'' - 4y' + y = 3r^2 e^{rx} - 4re^{rx} + e^{rx} = 0$$

$$\underbrace{(3r^2 - 4r + 1)}_{=0}\underbrace{e^{rx}}_{\neq 0} = 0$$

CHARACTERISTIC POLYNOMIAL IN r

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{16 - 12}}{6}$$

$$= \frac{4 \pm 2}{6}$$

$$= 1, \tfrac{1}{3} \rightarrow y_1 = e^x, \; y_2 = e^{\frac{1}{3}x}$$

$y = c_1 e^x + c_2 e^{\frac{1}{3}x}$ IS THE GENERAL SOL'N OF $3y'' - 4y' + 2y = 0$

$$W[e^x, e^{\frac{1}{3}x}] = \begin{vmatrix} e^x & e^{\frac{1}{3}x} \\ e^x & \frac{1}{3}e^{\frac{1}{3}x} \end{vmatrix} = \frac{1}{3}e^{\frac{4}{3}x} - e^{\frac{4}{3}x} = -\frac{2}{3}e^{\frac{4}{3}x} \neq 0$$

YOURSELF: PROVE THAT CHAR. POLY. OF

$$ay''+by'+cy=0$$

IS $ar^2+br+c=0$

SOLVE $y''+4y'+4y=0$ GUESS $y=e^{rx}$

$$r^2+4r+4=0$$

$$(r+2)^2=0$$

$$r=-2,-2$$

$$y_1=e^{-2x},\quad y_2=e^{-2x}$$

$$y=c_1e^{-2x}+c_2e^{-2x}=(c_1+c_2)e^{-2x}=Ce^{-2x}$$

IS NOT THE GENERAL SOL'N

$$W[e^{-2x},e^{-2x}] = \begin{vmatrix} e^{-2x} & e^{-2x} \\ -2e^{-2x} & -2e^{-2x} \end{vmatrix}$$

$$= e^{-2x}\cdot(-2e^{-2x})-(-2e^{-2x})e^{-2x}$$

$$=0$$

GUESS $y = v(x)e^{-2x} = ve^{-2x}$

$$y' = v'e^{-2x} - 2ve^{-2x}$$

$$y'' = v''e^{-2x} - 2v'e^{-2x}$$

$$-2v'e^{-2x} + 4ve^{-2x}$$

$$= v''e^{-2x} - 4v'e^{-2x} + 4ve^{-2x}$$

$$y'' + 4y' + 4y =$$